ON THE PROBLEM OF A STATIC ELASTIC CIRCULAR CYLINDRICAL SHELL WITH INITIAL DEFLECTION

(ODNA ZADACHA STATIKI UPRUGOI KRUGOVOI TSILINDRICHESKOI OBOLOCHKI, IMEIUSHCHEI NACHAL'NUIU POGIB')

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V. S. IVANOV
(Leningrad)

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1. The study of compressed cylindrical shells with an initial deviation from the circular shape has been the subject of quite a number of investigations. They are all, however, based upon the assumption of precise coincidence of the form of the additional elastic deflection with that of the intial deflection. The present paper considers the case of a shell, under the action of uniformly distributed external pressure, assuming deflectional equilibrium shapes characterized by a number of waves differing from that of the cyclical initial deflection. The solution of the problem is obtained in a non-linear treatment for loosely supported short cylindrical shells.

2. The following notations will be used in this paper: δ, r, l are the thickness, radius and length, respectively, of the shell; $\xi = x/r$, $\phi = y/r$ are dimensionless coordinates; f is the elastic deflection corresponding to the new wave formation; $a = W/r$ is the dimensionless amplitude of the deflection of the shell corresponding to the new wave formation; f^0 is the initial deflection; $a^0 = W^0/r$ is the dimensionless amplitude of the initial deflection; f_+ is the additional elastic deflection following the shape of the initial deflection; $a_+ = W_+/r$ is the dimensionless amplitude of the additional elastic deflection following the shape of the initial deflection; m is the number of waves of the initial deflection; n is the number of waves of the new wave formation; n_{*1} and n_{*2} are the numbers of waves appearing on the surface of the idealized shell at the upper and the lower critical pressures respectively; k^0 and k are the parameters of the form of the original deflection and of the new wave formation respectively; E and μ are Young's modulus and Poisson's ratio respectively; q is the external pressure,

$$\alpha = \frac{\pi r}{l}, \quad \gamma_1 = \frac{m}{\alpha}, \quad \gamma = \frac{n}{\alpha}, \quad D = \frac{1}{12(1-\mu^2)}\frac{\delta^2}{r^2}, \quad \nu^2 = \alpha^4 D$$

$$T_1{}^0 = -\frac{1}{2}qr, \quad T_2{}^0 = -qr, \quad f_0 = f^\circ + f_+, \quad a_0 = a^\circ + a_+$$

3. The equilibrium equation of a shell element at an additional deflection of arbitrary form becomes

$$\frac{\partial^2(F_0+F_+)}{\partial\varphi^2}\frac{\partial^2(f_0+f)}{\partial\xi^2} + \frac{\partial^2(F_0+F_+)}{\partial\xi^2}\frac{\partial^2(f_0+f)}{\partial\varphi^2} - $$
$$-2\frac{\partial^2(F_0+F_+)}{\partial\xi\,\partial\varphi}\frac{\partial^2(f_0+f)}{\partial\xi\,\partial\varphi} + \frac{\partial^2(F_0+F_+)}{\partial\xi^2} - D\nabla^2\nabla^2(f_+ + f) +$$
$$+\frac{T_1{}^0}{E\delta}\frac{\partial^2(f_0+f)}{\partial\xi^2} + \frac{T_2{}^0}{E\delta}\frac{\partial^2(f_0+f)}{\partial\varphi^2} = 0 \quad (1)$$

where F_0 is the value of the stress function, at the beginning of the new wave formation on the shell, for stresses produced by the combined bending following the form of the initial deflection, while F_+ is the stress function for the stresses which arise additionally at the middle surface of the shell in the process of deformation into the new wave form.

The functions F_0, F_+, f_0 and f, occurring in equation (1), are connected with each other by the additional conditions

$$\frac{\partial^2 F_0}{\partial\varphi^2}\frac{\partial^2 f_0}{\partial\xi^2} + \frac{\partial^2 F_0}{\partial\xi^2}\frac{\partial^2 f_0}{\partial\varphi^2} - 2\frac{\partial^2 F_0}{\partial\xi\,\partial\varphi}\frac{\partial^2 f_0}{\partial\varphi^2} + \frac{\partial^2 F_0}{\partial\xi^2} -$$
$$- D\nabla^2\nabla^2 f_+ + \frac{T_1{}^\circ}{E\delta}\frac{\partial^2 f_0}{\partial\xi^2} + \frac{T_2{}^\circ}{E\delta}\frac{\partial^2 f_0}{\partial\varphi^2} = 0 \quad (2)$$

$$\nabla^2\nabla^2 F_0 = \left(\frac{\partial^2 f_0}{\partial\xi\,\partial\varphi}\right)^2 - \frac{\partial^2 f_0}{\partial\xi^2}\frac{\partial^2 f_0}{\partial\varphi^2}\frac{\partial^2 f_0}{\partial\varphi^2} - \frac{\partial^2 f_0}{\partial\xi^2} -$$
$$-\left(\frac{\partial^2 f^0}{\partial\xi\,\partial\varphi}\right)^2 + \frac{\partial^2 f^0}{\partial\xi^2}\frac{\partial^2 f^0}{\partial\varphi^2} + \frac{\partial^2 f^0}{\partial\xi^2} \quad (3)$$

$$\nabla^2\nabla^2 F_+ = \left[\frac{\partial^2(f_0+f)}{\partial\xi\,\partial\varphi}\right]^2 - \frac{\partial^2(f_0+f)}{\partial\xi^2}\frac{\partial^2(f_0+f)}{\partial\varphi^2} - \frac{\partial^2(f_0+f)}{\partial\xi^2} - \left(\frac{\partial^2 f_0}{\partial\xi\,\partial\varphi}\right)^2 +$$
$$+\frac{\partial^2 f_0}{\partial\xi^2}\frac{\partial^2 f_0}{\partial\varphi^2} + \frac{\partial^2 f_0}{\partial\xi^2} \quad (4)$$

For the shell deflections we assume the following expressions

$$f^\circ = a^\circ(\sin m\varphi + k^\circ \sin^2 m\varphi)\sin\alpha\xi$$
$$f_+ = a_+(\sin m\varphi + k^\circ \sin^2 m\varphi)\sin\alpha\xi \quad (5)$$
$$f = a\ (\sin n\varphi + k\sin^2 n\varphi)\sin\alpha\xi$$

Having integrated the system of equations (1)-(4) by the procedure usually applied to the solution of nonlinear stability problems of idealized shells, we obtain the condition of equilibrium taking place after the beginning of deformation for *n* waves:

$$ea^3 - a^2(d - I_1 a_0) + a(c - a_0 L_1 + a_0^2 L_2) - \frac{qr}{E\delta}\alpha^2 h a = 0 \tag{6}$$

Here

$$\begin{aligned} e &= \alpha^4(k^4 e_1 + k^2 e_2 + e_3), & I_1 &= \alpha^2 k^{\circ}_1 (k^3 I_{11} + k I_{12}) \\ d &= \alpha^2 (k^3 d_1 + k d_2), & L_1 &= \alpha^2 k_0 (k^2 L_{11} + L_{12}) \\ c &= k^2(c_1 + \nu^2 c_2) + c_3 + \nu^2 c_4, & L_2 &= \alpha^4 [k^{02}(k^2 L_{21} + L_{23}) + k^2 L_{22} + L_{24}] \\ h &= k^2 h_1 + h_2 \end{aligned} \tag{7}$$

where

$$h_1 = 0.187 + 0.500\gamma^2, \qquad c_1 = 0.25 + \frac{1}{8(1+4\gamma^2)^2}, \qquad c_3 = \frac{1}{2(1+\gamma^2)^2}$$

$$h_2 = 0.25 + 0.50\gamma^2, \qquad c_2 = 0.375 + \gamma^2 + 2\gamma^4, \qquad c_4 = 0.50(1+\gamma^2)^2$$

$$d_1 = \frac{\gamma^2}{\pi}\left[0.75 + \frac{1}{24(1+\gamma^2)^2} + \frac{1.33}{(1+4\gamma^2)^2}\right], \qquad e_1 = \frac{\gamma^4}{32}\left[\left(1 + \frac{1}{\gamma^4}\right) + \frac{1}{2(1+\gamma^2)^2}\right]$$

$$d_2 = \frac{\gamma^2}{\pi}\left[0.75 + \frac{3.33}{(1+\gamma^2)^2} + \frac{3.66}{(4+\gamma^2)^2} + \frac{1}{(1+4\gamma^2)^2}\right], \qquad e_3 = \frac{\gamma^4}{32}\left(1 + \frac{1}{\gamma^4}\right)$$

$$e_2 = \frac{\gamma^4}{32}\left[\frac{60.5}{(4+\gamma^2)^2} + \frac{10}{\gamma^4} + \frac{1}{2(4+9\gamma^2)^2} + 2\left(1 + \frac{1}{\gamma^4}\right)\right] \tag{8}$$

$$I_{11} = \frac{1}{32}\left[\frac{3}{2} + \frac{11}{4}\frac{\gamma^4}{(1+\gamma^2)^2}\right], \qquad L_{11} = \frac{\gamma^2}{32}\left[\frac{a_+}{a_0} + \frac{1}{8(1+\gamma^2)^2} + \frac{2}{(1+4\gamma^2)^2}\right]$$

$$I_{12} = \frac{1}{32}\left[\frac{21}{2} + \frac{2\gamma^4}{(1+\gamma^2)^2} + \frac{33}{2}\frac{\gamma^4}{(\gamma^2+4)^2}\right]$$

$$L_{12} = \frac{\gamma^2}{32}\left[\frac{a_+}{a_0} + \frac{2}{(1+\gamma^2)^2} + \frac{2}{(4+\gamma^2)^2}\right]$$

$$L_{21} = \frac{1}{32}\left[\frac{5}{4} + \left(1 - \frac{a_+^2}{a_0^2}\right)\frac{\gamma^2\gamma_1^2}{2} + \frac{\gamma^4}{4(1+\gamma^2)^2} + \frac{\gamma_1^4}{4(1+\gamma_1^2)^2} + \right.$$
$$\left. + \frac{(\gamma_1+\gamma)^4}{16[(\gamma-\gamma_1)^2+1]^2} + \frac{(\gamma-\gamma_1)^4}{16[(\gamma+\gamma_1)^2+1]^2}\right]$$

$$L_{22} = \frac{1}{32}\left[3 + \left(1 - \frac{a_+^2}{a_0^2}\right)\frac{\gamma^2\gamma_1^2}{2} + \frac{\gamma_1^4}{(4+\gamma_1^2)^2} + \frac{(2\gamma+\gamma_1)^4}{4[(2\gamma-\gamma_1)^2+4]^2} + \frac{(2\gamma-\gamma_1)^4}{4[(2\gamma+\gamma_1)^2+4]^2}\right]$$

$$L_{23} = \frac{1}{32}\left[3 + \left(1 - \frac{a_+^2}{a_0^2}\right)\frac{\gamma^2\gamma_1^2}{2} + \frac{\gamma^4}{(\gamma^2-4)^2} + \frac{(\gamma+2\gamma_1)^4}{4[(\gamma-2\gamma_1)^2+4]^2} + \frac{(\gamma-2\gamma_1)^4}{4[(\gamma+2\gamma_1)^2+4]^2}\right]$$

$$L_{24} = \frac{1}{32}\left[4 + \left(1 - \frac{a_+^2}{a_0^2}\right)\frac{\gamma^2\gamma_1^2}{2} + \frac{(\gamma+\gamma_1)^4}{[(\gamma-\gamma_1)^2+4]^2} + \frac{(\gamma-\gamma_1)^4}{[(\gamma-\gamma_1)^2+4]^2}\right]$$

We note that the equation obtained is not valid when $m = n$, because its derivation is based upon the assumption that $m \neq n$, an assumption which considerably simplifies the computations.

The equation (6) is structurally analogous to the corresponding equation of the idealized shell. We can, therefore, state that the change of the wave formation on the surface of the shell is analogous to the loss of stability of the deflectionless equilibrium form of the idealized shell; in other words, this change of wave formation also takes place with "snap-through".

The difference is in the absolute amount of the critical pressures.

The upper critical pressure, from equation (6),

$$q_*^+ = \frac{E\delta}{r}\,\frac{1}{\alpha^2}\,\frac{c - a_0 L_1 + a_0^2 L_2}{h} \tag{9}$$

is characterized by a change in the wave formation at infinitesimal disturbances. The lower critical pressure

$$q_*^- = \frac{E\delta}{r}\,\frac{1}{\alpha^2}\,\frac{1}{h}\left[c - a_0 L_1 + a_0^2 L_2 - \frac{(d - a_0 l_1)^2}{4e}\right] \tag{10}$$

produces a change in the wave formation at finite disturbances.

If $a_0 = 0$ and $k = 0$, then equation (9) becomes identical with the known formula of Mises[2].

4. An unknown quantity appearing in formulas (9) and (10) is the magnitude of the amplitude of the additional deflection assumed by the shell at the moment of the beginning of the change in the wave formation.

We obtain an equation for the determination of a_+ from the system (2) and (3), with (5) taken into account:

$$ea_+^3 - a_+(d - 3ea^\circ) + a_+(c - a^\circ\overline{d + g} + 2ea^{\circ 2}) - \frac{qr}{E\delta}\,\alpha^2 h\,(a^\circ + a_+) = 0 \tag{11}$$

where e, d, c, h are the same as before, with γ replaced, however, by γ_1 and k by k^0;

$$g = \alpha^2 (k^{\circ 3} g_1 + k^\circ g_2)$$

where

$$g_1 = \frac{\gamma^2}{\pi}\left[\frac{1}{12} + \frac{1}{24(1+\gamma_1^2)^2}\right], \qquad g_2 = \frac{\gamma^2}{\pi}\left[\frac{1}{12} + \frac{3.66}{(4+\gamma^2)^2}\right]$$

Equation (11) has no zero roots, which indicates the presence of deflectional deformations at any arbitrarily small pressures. We note that if the amplitude of the initial deflection is smaller than

$$a^\circ < \frac{d^3}{27e^2c} = a_*^\circ$$

the equilibrium curve, defined by equation (11), is characterized by unstable states of equilibrium: the shell can undergo a "snap-through" without any change in the wave formation.

The equilibrium curve corresponding to the deformation following the form of the initial deflection, envelops the curves defined by equation (11) and corresponding to the values of the parameter k^0 from the side of the smallest pressures. Computations show that the quantity a_+ depends essentially on the number of the waves of the initial deflection, and that it increases rapidly when m approaches the values of n_{*1} and n_{*2}.

5. The Table presented below gives the results of the numerical computation of the critical pressures for a shell characterized by the ratios $r/l = 3$, $100\,\delta/r = 0.65$ ($\nu^2 = 0.030$) for three values of the wave numbers of the initial deflection at an amplitude of the latter equal to $a^0 = 0.001$ ($W^0/\delta = 0.15$); it has been assumed, in the computation of the lower (q_*^-) and the upper (q_*^+) critical pressures, that all three shells were given an additional elastic deflection following the form of the initial deflection and corresponding to a pressure of 24 atm, in other words, that in the process of precritical bending to m waves each shell could withstand 24 atm. Moreover, the surface of the shell was given a deviation to n waves with infinitely small or finite amplitude. The upper and lower critical pressures given in the Table represent the loading able to keep the prescribed disturbance in equilibrium.

TABLE

m	a^0	a_+	k^0	q	q^-_*	q^+_*	n_{*1}	n_{*2}, k
9	0.001	0.0035	0.4	24.0	16.6	22.5	14	12) 0.6
13	0.001	0.0125	0.6	24.0	56.0	63.0	14	12) 0.6
18	0.001	0.0050	0.3	24.0	28.1	30.8	14	12) 0.6

We mention for comparison that the values of the upper and lower critical loadings for the same shell, but without initial deflection, are respectively $q_*^+ = 22$ atm (for $n = 14$), $q_*^- = 16.2$ atm (for $n = 12$, $k = 0.6$. The Table indicates higher critical pressures for the shell with initial deflection as compared with the idealized shell, particularly when the number of waves approaches n_{*1} and n_{*2}

In this case "snap-through" may be caused not by change of wave formation, but, as shown above, by the loss of stability of deformation following the shape of the initial deflection (if $a^0 < a^0_*$). This statement becomes clear, if one takes into consideration that the increase in the precritical deflection of the shell is most intensive in this case, so that much more energy is needed for the change of the wave formation than when the number of waves differs from n_{*1} and n_{*2} by 3 to 4 units. The Table shows e.g. that at $m = 9$ the shell will not be able to withstand a pressure of 24 atm with a deformation following the initial deflection; a change will take place much earlier for the 12 waves form, since not only the lower, but also the upper critical pressure is below 24 atm (22.5 atm).

The shell with $m = 18$ is near to stability loss with 12 waves. Thus, a proximity of the number of waves of the initial deflection to n_{*1} and n_{*2} causes a rise of the critical pressures, producing in turn a change of wave formation, which in general may eliminate the possibility of "snap-through" (if $a^0 > a^0_*$); nevertheless, such an initial feflection is the most dangerous one. Its negative influence consists in a rapid increase in the deflections formed in accordance with the initial deflection with increase in pressure. In the case of highly-stressed shells such deflections may lead to intensive plastic deformations as great as those produced by "snap-through" in an idealized shell.

It is interesting to note that an initial deflection with a number m of waves smaller than n_{*1} and n_{*2} has practically no effect on the critical pressures (if $m = 9$). This theoretical conclusion is confirmed by the following well-known experimental fact: when shells having initial fabricational deflection with two, three or four waves are being tested in tanks, the loss of local stability of the shell takes place at pressures not much different from those calculated on the assumption of dealing with an idealized shell. The influence of an initial deflection with a number of waves larger than n_{*1} and n_{*2} is more essential for the following reasons: (1) more intensive precritical bending of the shell; (2) increase in the coefficients L_{21} and L_{24}, indicating the necessity of spending more work for the change from short waves to long than vice versa.

6. The computation of the critical pressures from (8) and (9) requires minimization of these expressions with respect to the parameters γ and k; if the pressures are calculated from (4), the minimization must be carried out with respect to k. A more correct procedure is the minimization of the potential energy. In the case of short cylindrical shells ($r/l > 3$), the error incurred by application of the principle of minimum pressure is, so the author finds, relatively small, in view of the small amount of the deflections accompanying the change of the wave formation.

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